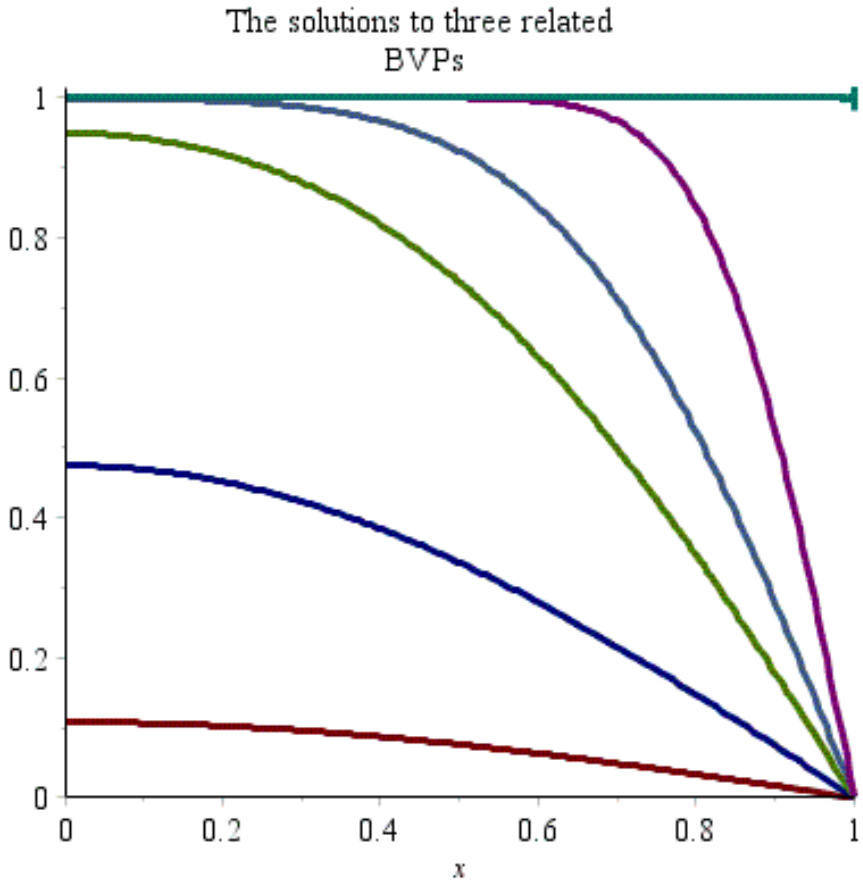
PEng-Consulting

Dynamic Analysis-Heat Equation

Algorithms for Parabolic Problems – Advantages and Disadvantages

REV 02



Uri Katanov

Pressure Vessel Eng., FEA and CFD Analyst

M.Sc.-MEng.

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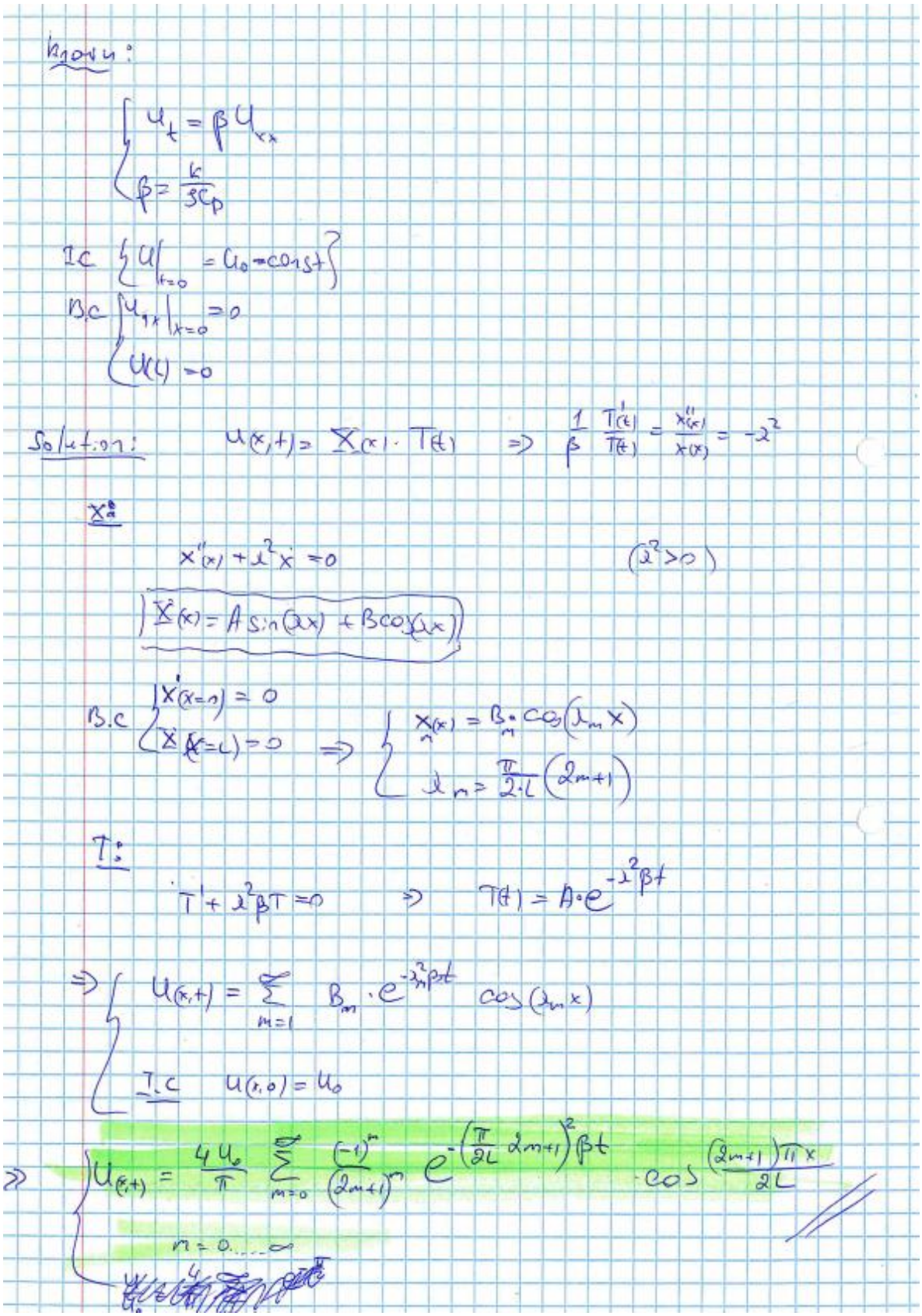
Exact solution



Maple programming

$$s1 := x \to sum\left(\left(\frac{4}{3.14^1}\cdot\frac{(-1)^m}{(2m+1)}\left(\frac{1}{\exp(1)^{\left(\frac{(3.14\cdot(2m+1))^2}{4}\right)\cdot 0.4}}\cdot\cos\left(\frac{3.14}{2}\cdot(2m+1)\right)*x\right)\right), m=0..10\right) : \% = value(\%);$$

$$\left(x\to\sum_{m=0}^{10}\frac{4(-1)^m\cos\left(\frac{3.14(2m+1)x}{2}\right)}{3.14(2m+1)(e)^{\frac{3.14^2(2m+1)^2\cdot 0.4}{4}}}\right) = \left(x\to\sum_{m=0}^{10}\frac{4(-1)^m\cos\left(\frac{3.14(2m+1)x}{2}\right)}{3.14(2m+1)(e)^{\frac{3.14^2(2m+1)^2\cdot 0.4}{4}}}\right)$$

>

>

$$s2 := x \to sum\left(\left(\frac{4}{3.14^1}\cdot\frac{(-1)^m}{(2m+1)}\left(\frac{1}{\exp(1)^{\left(\frac{(3.14\cdot(2m+1))^2}{4}\right)\cdot 0.1}}\cdot\cos\left(\frac{3.14}{2}\cdot(2m+1)\right)*x\right)\right), m=0..10\right) : \% = value(\%);$$

$$\left(x\to\sum_{m=0}^{10}\frac{4(-1)^m\cos\left(\frac{3.14(2m+1)x}{2}\right)}{3.14(2m+1)(e)^{\frac{3.14^2(2m+1)^2\cdot 0.1}{4}}}\right) = \left(x\to\sum_{m=0}^{10}\frac{4(-1)^m\cos\left(\frac{3.14(2m+1)x}{2}\right)}{3.14(2m+1)(e)^{\frac{3.14^2(2m+1)^2\cdot 0.1}{4}}}\right)$$

>

>

$$s3 := x \to sum\left(\left(\frac{4}{3.14^1}\cdot\frac{(-1)^m}{(2m+1)}\left(\frac{1}{\exp(1)^{\left(\frac{(3.14\cdot(2m+1))^2}{4}\right)\cdot 1}}\cdot\cos\left(\frac{3.14}{2}\cdot(2m+1)\right)*x\right)\right), m=0..10\right) : \% = value(\%);$$

$$\left(x \to \sum_{m=0}^{10} \frac{4\,(-1)^m \cos\left(\frac{3.14\,(2m+1)\,x}{2}\right)}{3.14\,(2m+1)\,(e)^{\frac{3.14^2\,(2m+1)^2}{4}}}\right) = \left(x \to \sum_{m=0}^{10} \frac{4\,(-1)^m \cos\left(\frac{3.14\,(2m+1)\,x}{2}\right)}{3.14\,(2m+1)\,(e)^{\frac{3.14^2\,(2m+1)^2}{4}}}\right)$$

> $s4 := x \to sum\left(\left(\frac{4}{3.14^1} \cdot \frac{(-1)^m}{(2m+1)}\left(\frac{1}{\exp(1)^{\left(\frac{(3.14\cdot(2m+1))^2}{4}\right)\cdot 0.04}} \cdot \cos\left(\frac{3.14}{2}\cdot(2m+1)*\mathrm{x}\right)\right)\right), m = 0..10\right) : \% = value(\%);$

$$\left(x \to \sum_{m=0}^{10} \frac{4\,(-1)^m \cos\left(\frac{3.14\,(2m+1)\,x}{2}\right)}{3.14\,(2m+1)\,(e)^{\frac{3.14^2\,(2m+1)^2\cdot 0.04}{4}}}\right) = \left(x \to \sum_{m=0}^{10} \frac{4\,(-1)^m \cos\left(\frac{3.14\,(2m+1)\,x}{2}\right)}{3.14\,(2m+1)\,(e)^{\frac{3.14^2\,(2m+1)^2\cdot 0.04}{4}}}\right)$$

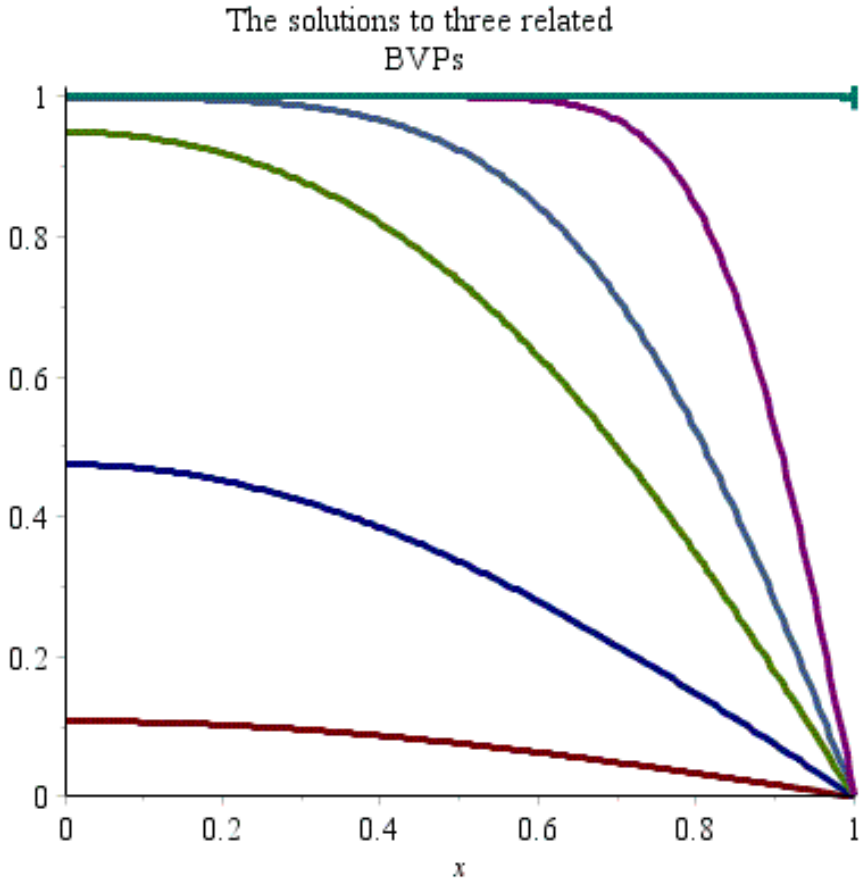
> $s5 := x \to sum\left(\left(\frac{4}{3.14^1} \cdot \frac{(-1)^m}{(2m+1)}\left(\frac{1}{\exp(1)^{\left(\frac{(3.14\cdot(2m+1))^2}{4}\right)\cdot 0.01}} \cdot \cos\left(\frac{3.14}{2}\cdot(2m+1)*\mathrm{x}\right)\right)\right), m = 0..10\right) : \% = value(\%);$

$$\left(x \to \sum_{m=0}^{10} \frac{4\,(-1)^m \cos\left(\frac{3.14\,(2m+1)\,x}{2}\right)}{3.14\,(2m+1)\,(e)^{\frac{3.14^2\,(2m+1)^2\cdot 0.01}{4}}}\right) = \left(x \to \sum_{m=0}^{10} \frac{4\,(-1)^m \cos\left(\frac{3.14\,(2m+1)\,x}{2}\right)}{3.14\,(2m+1)\,(e)^{\frac{3.14^2\,(2m+1)^2\cdot 0.01}{4}}}\right)$$

> $s6 := x \to sum\left(\left(\frac{4}{3.14^1} \cdot \frac{(-1)^m}{(2m+1)}\left(1 \cdot \cos\left(\frac{3.14}{2}\cdot(2m+1)\cdot \mathrm{x}\right)\right)\right), m = 0..20000\right) : \% = value(\%);$

$$\left(x \to \sum_{m=0}^{20000} \frac{4\,(-1)^m \cos\left(\frac{3.14\,(2m+1)\,x}{2}\right)}{3.14\,(2m+1)}\right) = \left(x \to \sum_{m=0}^{20000} \frac{4\,(-1)^m \cos\left(\frac{3.14\,(2m+1)\,x}{2}\right)}{3.14\,(2m+1)}\right)$$

> $plot(\{s1(x), s2(x), s3(x), s4(x), s5(x), s6(x)\}, x = 0..1, title = \text{"The solutions to three related BVPs"}, thickness = 3);$



FEM Approach Backward Euler

```
> #####Beuler-Final###
> with(LinearAlgebra) :
>
  beuler := proc(M, K, d0, n, dt, el, v0, a)
   local i, L, U, V ;
  U := Matrix(n + 1, 2, datatype = anything);
   V := Matrix(n + 1, 2, datatype = anything);
   L := M + dt·K;
  for i from 0 to n
   do
  U[1 + i, 1] := i·dt ;
   V[1 + i, 1] := i·dt;
   end do:
    U[1, 2] := v0;
   V[1, 2] := d0;
   for i from 1 to n
   do
  U[1 + i, 2] := (-1)·L^-1.K.(V[i, 2] + (1 − a)·dt·U[ i, 2]);
  V[1 + i, 2] := V[i, 2] + (1 − a)·dt·U[ i, 2] + a·dt·U[i + 1, 2] :
  end do:
  U;
  V;
  end proc;


beuler := proc(M, K, d0, n, dt, el, v0, a)
    local i, L, U, V;
    U := Matrix(n + 1, 2, datatype = anything);
    V := Matrix(n + 1, 2, datatype = anything);
    L := M + dt*K;
    for i from 0 to n do U[1 + i, 1] := i*dt; V[1 + i, 1] := i*dt end do;
    U[1, 2] := v0;
    V[1, 2] := d0;
    for i to n do
        U[1 + i, 2] := Typesetting:-delayDotProduct(Typesetting:-delayDotProduct( − 1/L,
        K), V[i, 2] + (1 − a)*dt*U[i, 2]);
        V[1 + i, 2] := V[i, 2] + (1 − a)*dt*U[i, 2] + a*dt*U[1 + i, 2]
    end do;
    U;
    V
end proc
```

BackWard Euler

> $a := 1$:

Inputs -- I.C. and B.C. --- Time steps=0.4

> $el := 5$:
> $h := Vector(el + 1, i \rightarrow (i-1)/el)$:
> $n := 2$:
> $Time := 0.8$:
> $dt := \left(\frac{\text{Time}}{\text{n}}\right) \cdot (1);$

$$dt := 0.4000000000$$

> $d0 := Vector(el + 1, i \rightarrow 1)$:

Global Mass and Stiffness Matrix

> $M := \left(\frac{\text{h}[2]}{6}\right) .Matrix(6, [[2, 1, 0, 0, 0, 0], [1, 4, 1, 0, 0, 0], [0, 1, 4, 1, 0, 0], [0, 0, 1, 4, 1, 0], [0, 0, 0, 1, 4, 1], [0, 0, 0, 0, 1, 4]])$:

> $K := Matrix(6, [[5, -5, 0, 0, 0, 0], [-5, 10, -5, 0, 0, 0], [0, -5, 10, -5, 0, 0], [0, 0, -5, 10, -5, 0], [0, 0, 0, -5, 10, -5], [0, 0, 0, 0, -5, 10]])$:

Calculations - FEM

> $v0 := -M^{-1}.K.d0$:

> $U := beuler(M, K, d0, n, dt, el, v0, a)$:
> $with(plots, pointplot)$:

> $Y := Vector(el + 1, U[\text{n} + 1, 2])$:
> $plot1 := pointplot(\langle h | Y \rangle, symbolsize = 15, color = ["Blue"])$:
> $pointplot(\langle h | Y \rangle, symbolsize = 15)$:

Calculations - Exact dt=0.4

>
>

$$Exact\_04 := x \rightarrow sum\left(\left(\frac{4}{3.14^{1}} \cdot \frac{(-1)^{m}}{(2\,\text{m}+1)^{1}}\left(\frac{1}{\exp(1)^{\left(\frac{(3.14\cdot(2\,\text{m}+1))^{2}}{4}\right)\cdot 0.4}} \cdot \cos\left(\frac{3.14}{2} \cdot (2\,\text{m}+1) * \text{x}\right)\right)\right), m = 0..500\right) : \% = value(\%);$$

$$\left(x \to \sum_{m=0}^{500} \frac{4\,(-1)^m \cos\left(\frac{3.14\,(2m+1)\,x}{2}\right)}{3.14\,(2m+1)\,(e)^{\frac{3.14^2\,(2m+1)^2 \cdot 0.4}{4}}}\right) = \left(x \to \sum_{m=0}^{500} \frac{4\,(-1)^m \cos\left(\frac{3.14\,(2m+1)\,x}{2}\right)}{3.14\,(2m+1)\,(e)^{\frac{3.14^2\,(2m+1)^2 \cdot 0.4}{4}}}\right)$$

```
> with(plots) :
> plot2 := plot(Exact_04(x), x = 0 ..1) :
> plot(Exact_04(x), x = 0 ..1) :
> display([plot1, plot2], ) :
```

Time steps = 0.1

```
> n := 2 :
> Time := 0.2 :
> dt := (Time/n)·(1);
```

$$dt := 0.1000000000$$

```
> U := beuler(M, K, d0, n, dt, el, v0, a) :
> with(plots, pointplot) :

> Y := Vector(el + 1, U[n + 1, 2]) :
> plot3 := pointplot(⟨h|Y⟩, symbolsize = 15, color = ["Blue"]) :
> pointplot(⟨h|Y⟩, symbolsize = 15) :
```

Calculations - Exact dt=0.1

```
>
> Exact_01 := x→ sum(((4/3.14^1)·((-1)^m/(2 m + 1)^1)(1/exp(1)^((3.14·(2 m + 1))^2/4)·0.1)·cos(3.14/2·(2 m + 1)*x))), m = 0 ..500) : %= value(%);
```

$$\left(x \to \sum_{m=0}^{500} \frac{4\,(-1)^m \cos\left(\frac{3.14\,(2m+1)\,x}{2}\right)}{3.14\,(2m+1)\,(e)^{\frac{3.14^2\,(2m+1)^2 \cdot 0.1}{4}}}\right) = \left(x \to \sum_{m=0}^{500} \frac{4\,(-1)^m \cos\left(\frac{3.14\,(2m+1)\,x}{2}\right)}{3.14\,(2m+1)\,(e)^{\frac{3.14^2\,(2m+1)^2 \cdot 0.1}{4}}}\right)$$

```
> with(plots) :
> plot4 := plot(Exact_01(x), x = 0..1) :
> plot(Exact_01(x), x = 0..1) :
```

Time steps = 1

```
> n := 2 :
> Time := 2 :
> dt := (Time/n)·(1);
```

$$dt := 1$$

```
> U := beuler(M, K, d0, n, dt, el, v0, a) :
> with(plots, pointplot) :

> Y := Vector(el + 1, U[n + 1, 2]) :
> plot5 := pointplot(⟨h|Y⟩, symbolsize = 15, color = ["Blue"]) :
> pointplot(⟨h|Y⟩, symbolsize = 15) :
```

Calculations - Exact dt=1

```
>
> Exact_1 := x→ sum((4/3.14^1)·((-1)^m/(2 m + 1)^1)(1/exp(1)^((3.14·(2 m + 1))^2/4)·1)·cos(3.14/2·(2 m + 1)*x)), m = 0..500) : %=value(%);
```

$$\left(x\to \sum_{m=0}^{500} \frac{4\,(-1)^m \cos\left(\frac{3.14\,(2m+1)\,x}{2}\right)}{3.14\,(2m+1)\,(e)^{\frac{3.14^2\,(2m+1)^2}{4}}}\right) = \left(x\to \sum_{m=0}^{500} \frac{4\,(-1)^m \cos\left(\frac{3.14\,(2m+1)\,x}{2}\right)}{3.14\,(2m+1)\,(e)^{\frac{3.14^2\,(2m+1)^2}{4}}}\right)$$

```
> with(plots) :
> plot6 := plot(Exact_1(x), x = 0..1) :
> plot(Exact_1(x), x = 0..1) :
```

Time steps = 0.04

```
> n := 2 :
> Time := 0.08 :
> dt := (Time/n)·(1);
```

$$dt := 0.04000000000$$

```
> U := beuler(M, K, d0, n, dt, el, v0, a) :
> with(plots, pointplot) :
```

```
> Y := Vector(el + 1, U[n + 1, 2]) :
> plot7 := pointplot(⟨h|Y⟩, symbolsize = 15, color = ["Blue"]) :
> pointplot(⟨h|Y⟩, symbolsize = 15, color = ["DarkGreen"]) :
```

Calculations - Exact dt=0.04

```
> >
```

$$Exact\_004 := x \to sum\left(\left(\left(\frac{4}{3.14^{1}}\cdot\frac{(-1)^{m}}{(2\,m+1)^{1}}\left(\frac{1}{\exp(1)^{\left(\frac{(3.14\cdot(2\,m+1))^{2}}{4}\right)\cdot 0.04}}\cdot\cos\left(\frac{3.14}{2}\cdot(2\,m+1)*x\right)\right)\right),\ m=0..500\right)\right) : \% = value(\%);$$

$$\left(x\to\sum_{m=0}^{500}\frac{4\,(-1)^{m}\cos\left(\frac{3.14\,(2\,m+1)\,x}{2}\right)}{3.14\,(2\,m+1)\,(e)^{\frac{3.14^{2}\,(2\,m+1)^{2}\cdot 0.04}{4}}}\right)=\left(x\to\sum_{m=0}^{500}\frac{4\,(-1)^{m}\cos\left(\frac{3.14\,(2\,m+1)\,x}{2}\right)}{3.14\,(2\,m+1)\,(e)^{\frac{3.14^{2}\,(2\,m+1)^{2}\cdot 0.04}{4}}}\right)$$

```
> with(plots) :
> plot8 := plot(Exact_004(x), x = 0..1) :
> plot(Exact_004(x), x = 0..1) :
```

Time steps = 0.01

```
> n := 2 :
> Time := 0.02 :
> dt := (Time/n)·(1);
```

$$dt := 0.01000000000$$

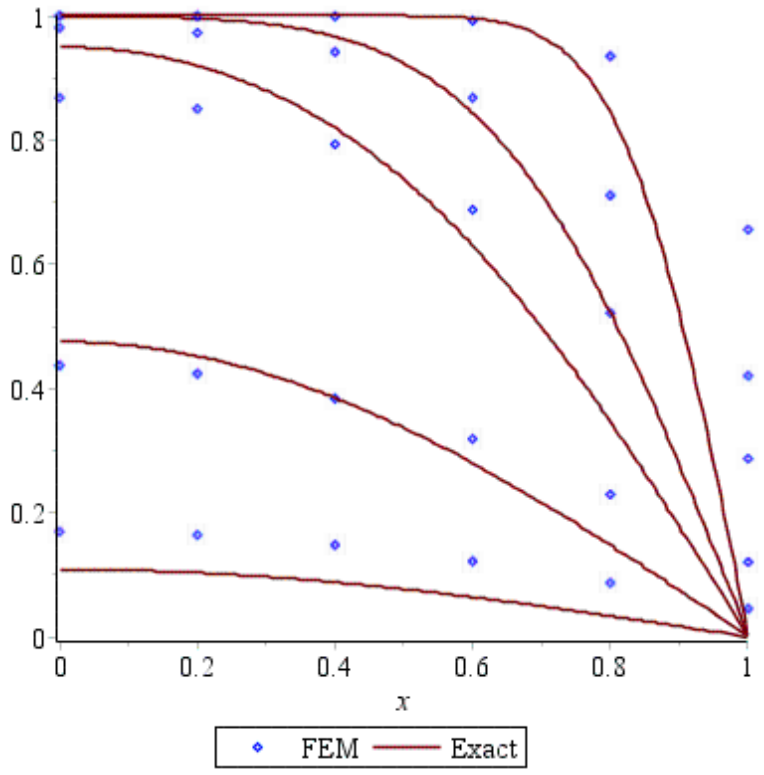
```
> U := beuler(M, K, d0, n, dt, el, v0, a) :
> with(plots, pointplot) :

> Y := Vector(el + 1, U[n + 1, 2]) :
> plot9 := pointplot(⟨h|Y⟩, symbolsize = 15, color = ["Blue"]) :
> pointplot(⟨h|Y⟩, symbolsize = 15, color = ["DarkGreen"]) :
```

Calculations - Exact dt=0.01

$$Exact\_001 := x \rightarrow sum\left(\left(\frac{4}{3.14^{1}} \cdot \frac{(-1)^{m}}{(2\,m+1)^{1}}\left(\frac{1}{\exp(1)^{\left(\frac{(3.14\cdot(2\,m+1))^{2}}{4}\right)\cdot 0.01}} \cdot \cos\left(\frac{3.14}{2}\cdot(2\,m+1)*x\right)\right)\right), m = 0..500\right) : \% = value(\%);$$

$$\left(x \rightarrow \sum_{m=0}^{500} \frac{4\,(-1)^{m}\cos\left(\frac{3.14\,(2\,m+1)\,x}{2}\right)}{3.14\,(2\,m+1)\,(e)^{\frac{3.14^{2}\,(2\,m+1)^{2}\cdot 0.01}{4}}}\right) = \left(x \rightarrow \sum_{m=0}^{500} \frac{4\,(-1)^{m}\cos\left(\frac{3.14\,(2\,m+1)\,x}{2}\right)}{3.14\,(2\,m+1)\,(e)^{\frac{3.14^{2}\,(2\,m+1)^{2}\cdot 0.01}{4}}}\right)$$

```
> with(plots) :
> plot10 := plot(Exact_001(x), x = 0..1) :
> plot(Exact_001(x), x = 0..1) :
> display([plot1, plot2, plot3, plot4, plot5, plot6, plot7, plot8, plot9, plot10]);
```



FEM Approach Trapezoidal Rule

```
> #####Ceuler-Final###
> with(LinearAlgebra) :
>
 beuler := proc(M, K, d0, n, dt, el, v0, a)
  local i, L, U, V ;
 U := Matrix(n + 1, 2, datatype = anything);
  V := Matrix(n + 1, 2, datatype = anything);
  L := M + dt·K;
 for i from 0 to n
  do
 U[1 + i, 1] := i·dt ;
  V[1 + i, 1] := i·dt;
 end do:
  U[1, 2] := v0;
 V[1, 2] := d0;
 for i from 1 to n
 do
 U[1 + i, 2] := (-1)·L^(-1).K.(V[i, 2] + (1 − a)·dt·U[ i, 2]);
 V[1 + i, 2] := V[i, 2] + (1 − a)·dt·U[ i, 2] + a·dt·U[i + 1, 2] :
 end do:
 U;
 V;
 end proc;


beuler := proc(M, K, d0, n, dt, el, v0, a)
    local i, L, U, V;
    U := Matrix(n + 1, 2, datatype = anything);
    V := Matrix(n + 1, 2, datatype = anything);
    L := M + dt*K;
    for i from 0 to n do U[1 + i, 1] := i*dt; V[1 + i, 1] := i*dt end do;
    U[1, 2] := v0;
    V[1, 2] := d0;
    for i to n do
        U[1 + i, 2] := Typesetting:-delayDotProduct(Typesetting:-delayDotProduct( − 1/L,
        K), V[i, 2] + (1 − a)*dt*U[i, 2]);
        V[1 + i, 2] := V[i, 2] + (1 − a)*dt*U[i, 2] + a*dt*U[1 + i, 2]
    end do;
    U;
    V
end proc
```

Trapezoidal Rule

> $a := \frac{1}{2}$:

Inputs -- I.C. and B.C. --- Time steps=0.4

> $el := 5$:
> $h := Vector(el + 1, i \to (i-1)/el)$:
> $n := 2$:
> $Time := 0.8$:
> $dt := \left(\frac{\text{Time}}{\text{n}}\right) \cdot (1)$;

$$dt := 0.4000000000$$

> $d0 := Vector(el + 1, i \to 1)$:

Global Mass and Stiffness Matrix

> $M := \left(\frac{\text{h}[2]}{6}\right) Matrix(6, [[2, 1, 0, 0, 0, 0], [1, 4, 1, 0, 0, 0], [0, 1, 4, 1, 0, 0], [0, 0, 1, 4, 1, 0], [0, 0, 0, 1, 4, 1], [0, 0, 0, 0, 1, 4]])$:

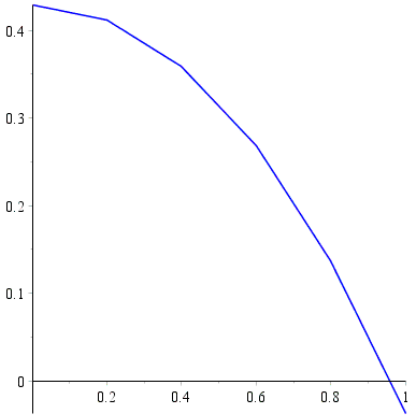
> $K := Matrix(6, [[5, -5, 0, 0, 0, 0], [-5, 10, -5, 0, 0, 0], [0, -5, 10, -5, 0, 0], [0, 0, -5, 10, -5, 0], [0, 0, 0, -5, 10, -5], [0, 0, 0, 0, -5, 10]])$:

Calculations - FEM

> $v0 := -M^{-1}.K.d0$:

> $U := beuler(M, K, d0, n, dt, el, v0, a)$:
> $with(plots, pointplot)$:

> $Y := Vector(el + 1, U[\text{n} + 1, 2])$:
> $plot1 := pointplot(\langle h|Y\rangle, symbolsize = 15, color = ["Blue"])$:
> $plot(\langle h|Y\rangle, symbolsize = 15, color = ["Blue"])$;



Calculations - Exact dt=0.4

>

> $$Exact\_04 := x \rightarrow sum\left(\left(\frac{4}{3.14^{1}} \cdot \frac{(-1)^{m}}{(2\,m+1)^{1}}\left(\frac{1}{\exp(1)^{\left(\frac{(3.14\cdot(2\,m+1))^{2}}{4}\right)\cdot 0.4}} \cdot \cos\left(\frac{3.14}{2}\cdot(2\,m+1)*x\right)\right)\right), m = 0..1\right) : \% = value(\%);$$

$$\left(x \rightarrow \sum_{m=0}^{1} \frac{4\,(-1)^{m}\cos\left(\frac{3.14\,(2\,m+1)\,x}{2}\right)}{3.14\,(2\,m+1)\,(e)^{\frac{3.14^{2}\,(2\,m+1)^{2}\cdot 0.4}{4}}}\right) = \left(x \rightarrow \sum_{m=0}^{1} \frac{4\,(-1)^{m}\cos\left(\frac{3.14\,(2\,m+1)\,x}{2}\right)}{3.14\,(2\,m+1)\,(e)^{\frac{3.14^{2}\,(2\,m+1)^{2}\cdot 0.4}{4}}}\right)$$

> $with(plots) :$
> $plot2 := plot(Exact\_04(x), x = 0..1) :$
> $plot(Exact\_04(x), x = 0..1) :$
> $display([plot1, plot2],) :$

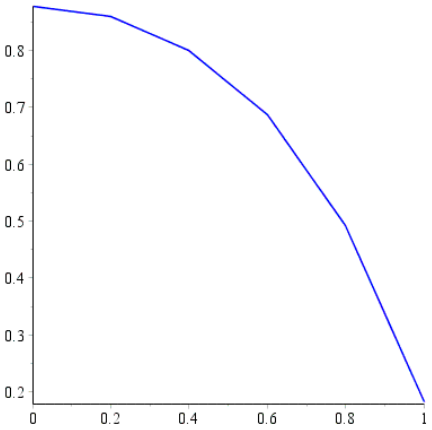
Time steps = 0.1

> $n := 2 :$
> $Time := 0.2 :$
> $dt := \left(\frac{\text{Time}}{\text{n}}\right)\cdot(1);$

$$dt := 0.1000000000$$

> $U := beuler(M, K, d0, n, dt, el, v0, a) :$
> $with(plots, pointplot) :$

> $Y := Vector(el + 1, U[\text{n} + 1, 2]) :$
> $plot3 := pointplot(\langle h|Y\rangle, symbolsize = 15, color = ["Blue"]) :$
> $plot(\langle h|Y\rangle, symbolsize = 15, color = ["Blue"]);$



Calculations - Exact dt=0.1

>
>
$$Exact\_01 := x \rightarrow sum\left(\left(\frac{4}{3.14^{1}}\cdot\frac{(-1)^{m}}{(2\,m+1)^{1}}\left(\frac{1}{\exp(1)^{\left(\frac{(3.14\cdot(2\,m+1))^{2}}{4}\right)\cdot 0.1}}\cdot\cos\left(\frac{3.14}{2}\cdot(2\,m+1)*x\right)\right)\right), m=0..1\right) : \%=value(\%);$$

$$\left(x\rightarrow\sum_{m=0}^{1}\frac{4\,(-1)^{m}\cos\left(\frac{3.14\,(2\,m+1)\,x}{2}\right)}{3.14\,(2\,m+1)\,(e)^{\frac{3.14^{2}\,(2\,m+1)^{2}\cdot 0.1}{4}}}\right)=\left(x\rightarrow\sum_{m=0}^{1}\frac{4\,(-1)^{m}\cos\left(\frac{3.14\,(2\,m+1)\,x}{2}\right)}{3.14\,(2\,m+1)\,(e)^{\frac{3.14^{2}\,(2\,m+1)^{2}\cdot 0.1}{4}}}\right)$$

> $with(plots) :$
> $plot4 := plot(Exact\_01(x), x=0..1) :$
> $plot(Exact\_01(x), x=0..1) :$

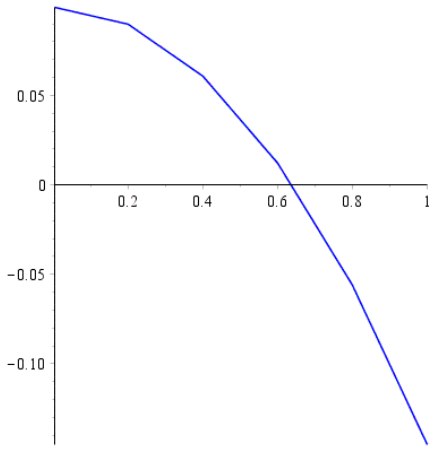
Time steps = 1

> $n := 2 :$
> $Time := 2 :$
> $dt := \left(\frac{\text{Time}}{\text{n}}\right)\cdot(1);$

$$dt := 1$$

> $U := beuler(M, K, d0, n, dt, el, v0, a) :$
> $with(plots, pointplot) :$

> $Y := Vector(el+1, U[\text{n}+1, 2]) :$
> $plot5 := pointplot(\langle h|Y\rangle, symbolsize=15, color=["Blue"]) :$
> $plot(\langle h|Y\rangle, symbolsize=15, color=["Blue"]);$



Calculations - Exact dt=1

>

> $Exact\_1 := x \rightarrow sum\left(\left(\frac{4}{3.14^1}\cdot\frac{(-1)^m}{(2\,m+1)^1}\left(\frac{1}{\exp(1)^{\left(\frac{(3.14\cdot(2\,m+1))^2}{4}\right)\cdot 1}}\cdot\cos\left(\frac{3.14}{2}\cdot(2\,m+1)*x\right)\right)\right), m=0..1\right) : \%=value(\%);$

$$\left(x\rightarrow\sum_{m=0}^{1}\frac{4\,(-1)^m\cos\left(\frac{3.14\,(2\,m+1)\,x}{2}\right)}{3.14\,(2\,m+1)\,(e)^{\frac{3.14^2\,(2\,m+1)^2}{4}}}\right)=\left(x\rightarrow\sum_{m=0}^{1}\frac{4\,(-1)^m\cos\left(\frac{3.14\,(2\,m+1)\,x}{2}\right)}{3.14\,(2\,m+1)\,(e)^{\frac{3.14^2\,(2\,m+1)^2}{4}}}\right)$$

> $with(plots) :$
> $plot6 := plot(Exact\_1(x), x=0..1) :$
> $plot(Exact\_1(x), x=0..1) :$

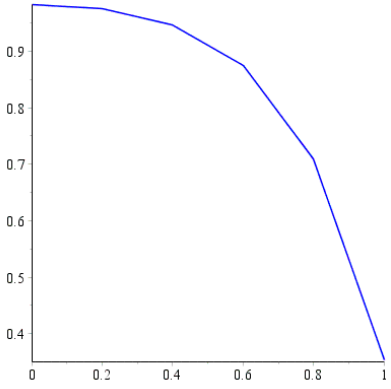
Time steps = 0.04

> $n := 2 :$
> $Time := 0.08 :$
> $dt := \left(\frac{\text{Time}}{\text{n}}\right)\cdot(1);$

$$dt := 0.04000000000$$

> $U := beuler(M, K, d0, n, dt, el, v0, a) :$
> $with(plots, pointplot) :$

> $Y := Vector(el+1, U[\text{n}+1, 2]) :$
> $plot7 := pointplot(\langle h|Y\rangle, symbolsize=15, color=["Blue"]) :$
> $plot(\langle h|Y\rangle, symbolsize=15, color=["Blue"]);$



Calculations - Exact dt=0.04

>

> $$Exact\_004 := x \rightarrow sum\left(\left(\frac{4}{3.14^{1}} \cdot \frac{(-1)^{m}}{(2\,m+1)^{1}}\left(\frac{1}{\exp(1)^{\left(\frac{(3.14\cdot(2\,m+1))^{2}}{4}\right)\cdot 0.04}} \cdot \cos\left(\frac{3.14}{2} \cdot (2\,m+1) * x\right)\right)\right), m = 0..1\right) : \% = value(\%);$$

$$\left(x \rightarrow \sum_{m=0}^{1} \frac{4\,(-1)^{m}\cos\left(\frac{3.14\,(2\,m+1)\,x}{2}\right)}{3.14\,(2\,m+1)\,(e)^{\frac{3.14^{2}\,(2\,m+1)^{2}\cdot 0.04}{4}}}\right) = \left(x \rightarrow \sum_{m=0}^{1} \frac{4\,(-1)^{m}\cos\left(\frac{3.14\,(2\,m+1)\,x}{2}\right)}{3.14\,(2\,m+1)\,(e)^{\frac{3.14^{2}\,(2\,m+1)^{2}\cdot 0.04}{4}}}\right)$$

> $with(plots) :$
> $plot8 := plot(Exact\_004(x), x = 0..1) :$
> $plot(Exact\_004(x), x = 0..1) :$

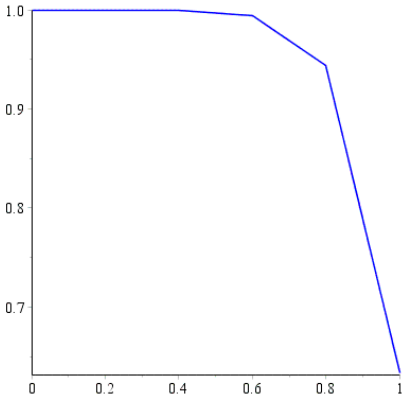
Time steps = 0.01

> $n := 2 :$
> $Time := 0.02 :$
> $dt := \left(\frac{\text{Time}}{\text{n}}\right) \cdot (1);$

$$dt := 0.01000000000$$

> $U := beuler(M, K, d0, n, dt, el, v0, a) :$
> $with(plots, pointplot) :$

> $Y := Vector(el + 1, U[\text{n} + 1, 2]) :$
> $plot9 := pointplot(\langle h|Y\rangle, symbolsize = 15, color = [\text{"Blue"}]) :$
> $plot(\langle h|Y\rangle, symbolsize = 15, color = [\text{"Blue"}]);$



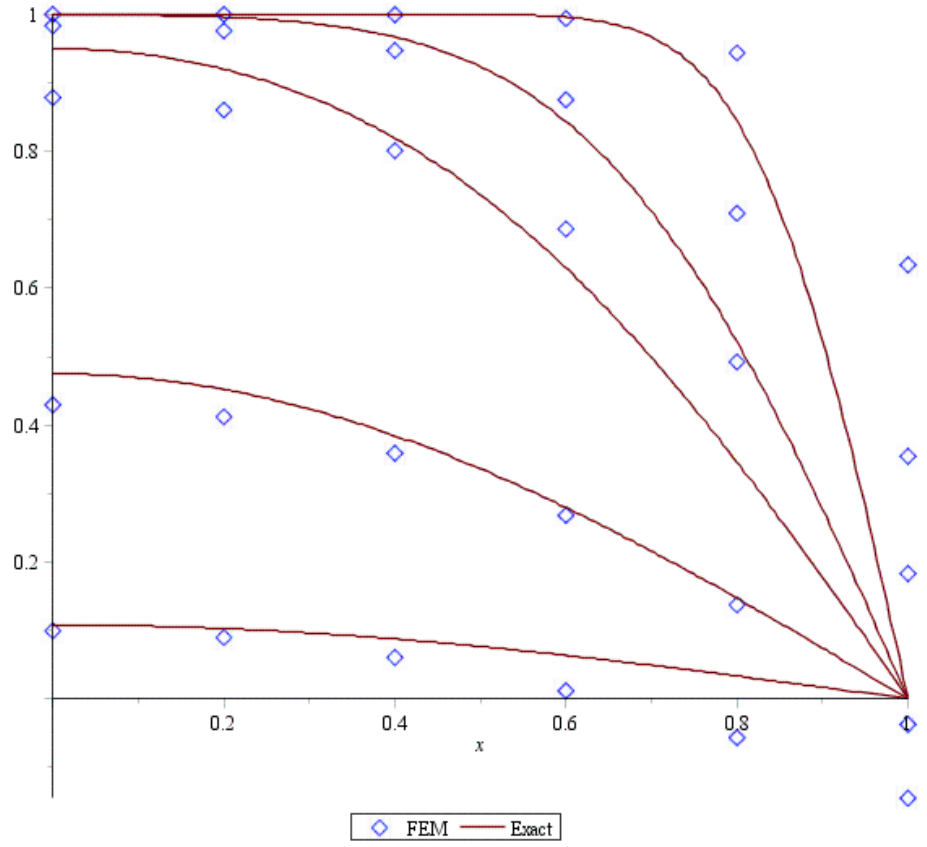
Calculations - Exact dt=0.01

>

> $$Exact\_001 := x \rightarrow sum\left(\left(\frac{4}{3.14^{1}} \cdot \frac{(-1)^{m}}{(2\,m+1)^{1}}\left(\frac{1}{\exp(1)^{\left(\frac{(3.14\cdot(2\,m+1))^{2}}{4}\right)\cdot 0.01}} \cdot \cos\left(\frac{3.14}{2} \cdot (2\,m+1) * x\right)\right)\right), m = 0..1\right) : \% = value(\%);$$

$$\left(x \rightarrow \sum_{m=0}^{1} \frac{4\,(-1)^{m}\cos\left(\frac{3.14\,(2\,m+1)\,x}{2}\right)}{3.14\,(2\,m+1)\,(e)^{\frac{3.14^{2}\,(2\,m+1)^{2}\cdot 0.01}{4}}}\right) = \left(x \rightarrow \sum_{m=0}^{1} \frac{4\,(-1)^{m}\cos\left(\frac{3.14\,(2\,m+1)\,x}{2}\right)}{3.14\,(2\,m+1)\,(e)^{\frac{3.14^{2}\,(2\,m+1)^{2}\cdot 0.01}{4}}}\right)$$

> $with(plots)$:
> $plot10 := plot(Exact\_001(x), x = 0..1)$:
> $plot(Exact\_001(x), x = 0..1)$:
> $display([plot1, plot2, plot3, plot4, plot5, plot6, plot7, plot8, plot9, plot10]);$
> $display([plot1, plot2, plot3, plot4, plot5, plot6, plot7, plot8, plot9, plot10]);$



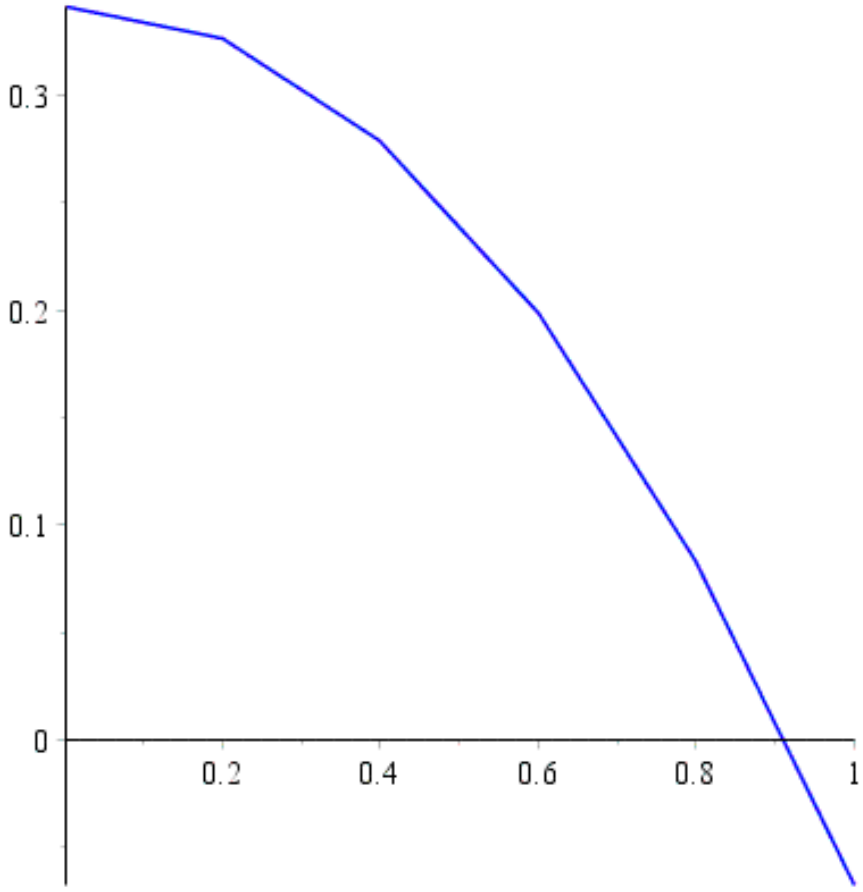
####### ***Time steps = 0.5*** #########

> $n := 2$:
> $Time := 1$:
> $dt := \left(\frac{\text{Time}}{\text{n}}\right) \cdot (1)$;

$$dt := \frac{1}{2}$$

> $U := beuler(M, K, d0, n, dt, el, v0, a)$:
> $with(plots, pointplot)$:

> $Y := Vector(el + 1, U[\text{n} + 1, 2])$:
> $plot11 := pointplot(\langle h|Y\rangle, symbolsize = 15, color = [\text{"Blue"}])$:
> $plot(\langle h|Y\rangle, symbolsize = 15, color = [\text{"Blue"}])$;



Calculations - Exact dt=0.5

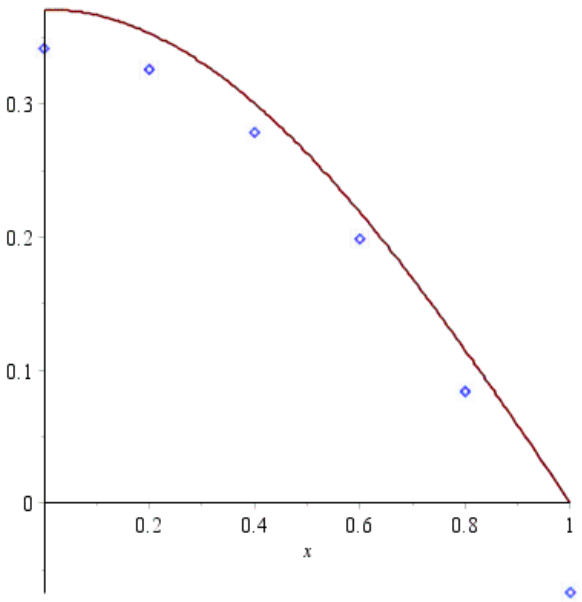
>

>

$$Exact\_05 := x \rightarrow sum\left(\left(\frac{4}{3.14^{1}} \cdot \frac{(-1)^{m}}{(2\,m+1)^{1}}\left(\frac{1}{\exp(1)^{\left(\frac{(3.14\cdot(2\,m+1))^{2}}{4}\right)\cdot 0.5}} \cdot \cos\left(\frac{3.14}{2} \cdot (2\,m+1) * x\right)\right)\right), m = 0..1\right) : \% = value(\%);$$

$$\left(x \rightarrow \sum_{m=0}^{1} \frac{4\,(-1)^{m}\cos\left(\frac{3.14\,(2\,m+1)\,x}{2}\right)}{3.14\,(2\,m+1)\,(e)^{\frac{3.14^{2}\,(2\,m+1)^{2}\cdot 0.5}{4}}}\right) = \left(x \rightarrow \sum_{m=0}^{1} \frac{4\,(-1)^{m}\cos\left(\frac{3.14\,(2\,m+1)\,x}{2}\right)}{3.14\,(2\,m+1)\,(e)^{\frac{3.14^{2}\,(2\,m+1)^{2}\cdot 0.5}{4}}}\right)$$

> $with(plots) :$
> $plot12 := plot(Exact\_05(x), x = 0..1) :$
> $plot(Exact\_05(x), x = 0..1) :$
> $display([plot11, plot12]);$



4.0-FEM Approach Forward Euler

```
> #####Feuler-Final###
> with(LinearAlgebra) :
>
  beuler := proc(M, K, d0, n, dt, el, v0, a)
   local i, L, U, V;
  U := Matrix(n + 1, 2, datatype = anything);
   V := Matrix(n + 1, 2, datatype = anything);
   L := M + dt·K;
  for i from 0 to n
   do
  U[1 + i, 1] := i·dt;
   V[1 + i, 1] := i·dt;
   end do:
    U[1, 2] := v0;
   V[1, 2] := d0;
   for i from 1 to n
   do
   U[1 + i, 2] := (-1)·L^(-1).K.(V[i, 2] + (1 − a)·dt·U[ i, 2]);
   V[1 + i, 2] := V[i, 2] + (1 − a)·dt·U[ i, 2] + a·dt·U[i + 1, 2] :
   end do:
   U;
   V;
   end proc;


  beuler := proc(M, K, d0, n, dt, el, v0, a)
      local i, L, U, V;
      U := Matrix(n + 1, 2, datatype = anything);
      V := Matrix(n + 1, 2, datatype = anything);
      L := M + dt*K;
      for i from 0 to n do U[1 + i, 1] := i*dt; V[1 + i, 1] := i*dt end do;
      U[1, 2] := v0;
      V[1, 2] := d0;
      for i to n do
          U[1 + i, 2] := Typesetting:-delayDotProduct(Typesetting:-delayDotProduct( − 1/L,
          K), V[i, 2] + (1 − a)*dt*U[i, 2]);
          V[1 + i, 2] := V[i, 2] + (1 − a)*dt*U[i, 2] + a*dt*U[1 + i, 2]
      end do;
      U;
      V
  end proc
```

Forward Euler

```
> a := 0 :
```

Inputs -- I.C. and B.C. --- Time steps=0.4

```
> el := 5 :
> h := Vector(el + 1, i→(i-1)/el) :
> n := 2 :
> Time := 0.8 :
> dt := (Time/n)·(1);
```

$$dt := 0.4000000000$$

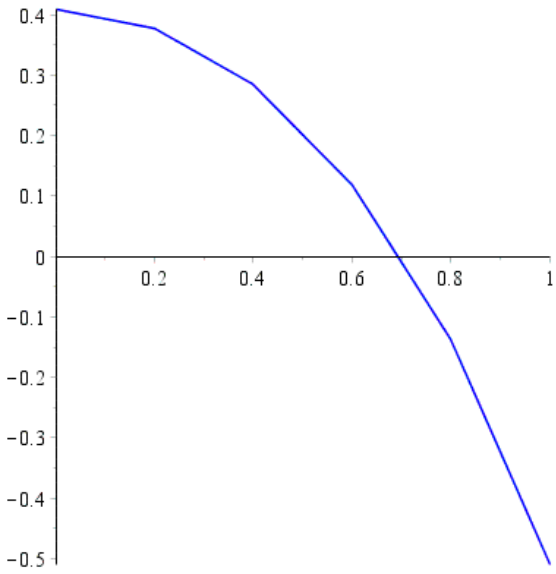
```
> d0 := Vector(el + 1, i→1) :
```

Global Mass and Stiffness Matrix

```
> M := (h[2]/6).Matrix(6, [[2, 1, 0, 0, 0, 0], [1, 4, 1, 0, 0, 0], [0, 1, 4, 1, 0, 0], [0, 0, 1, 4, 1, 0], [0,
      0, 0, 1, 4, 1], [0, 0, 0, 0, 1, 4]]) :
> K := Matrix(6, [[5,-5, 0, 0, 0, 0], [-5, 10,-5, 0, 0, 0], [0,-5, 10,-5, 0, 0], [0, 0,-5, 10,-5, 0],
      [0, 0, 0,-5, 10,-5], [0, 0, 0, 0,-5, 10]]) :
```

Calculations - FEM

```
> v0 := -M^-1.K.d0 :
> U := beuler(M, K, d0, n, dt, el, v0, a) :
> with(plots, pointplot) :
> Y := Vector(el + 1, U[n + 1, 2]) :
> plot1 := pointplot(⟨h|Y⟩, symbolsize = 15, color = ["Blue"]) :
> plot(⟨h|Y⟩, symbolsize = 15, color = ["Blue"]);
```



Calculations - Exact dt=0.4

>

> $$Exact\_04 := x \rightarrow sum\left(\left(\frac{4}{3.14^{1}} \cdot \frac{(-1)^{m}}{(2\,m+1)^{1}}\left(\frac{1}{\exp(1)^{\left(\frac{(3.14\cdot(2\,m+1))^{2}}{4}\right)\cdot 0.4}} \cdot \cos\left(\frac{3.14}{2} \cdot (2\,m+1) * x\right)\right)\right), m = 0..1\right) : \% = value(\%);$$

$$\left(x \rightarrow \sum_{m=0}^{1} \frac{4\,(-1)^{m}\cos\left(\frac{3.14\,(2\,m+1)\,x}{2}\right)}{3.14\,(2\,m+1)\,(e)^{\frac{3.14^{2}\,(2\,m+1)^{2}\cdot 0.4}{4}}}\right) = \left(x \rightarrow \sum_{m=0}^{1} \frac{4\,(-1)^{m}\cos\left(\frac{3.14\,(2\,m+1)\,x}{2}\right)}{3.14\,(2\,m+1)\,(e)^{\frac{3.14^{2}\,(2\,m+1)^{2}\cdot 0.4}{4}}}\right)$$

> $with(plots) :$
> $plot2 := plot(Exact\_04(x), x = 0..1) :$
> $plot(Exact\_04(x), x = 0..1) :$
> $display([plot1, plot2],) :$

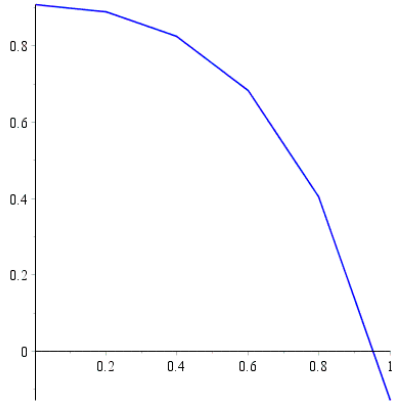
Time steps = 0.1

> $n := 2 :$
> $Time := 0.2 :$
> $dt := \left(\frac{\text{Time}}{\text{n}}\right) \cdot (1);$

$$dt := 0.1000000000$$

> $U := beuler(M, K, d0, n, dt, el, v0, a) :$
> $with(plots, pointplot) :$

> $Y := Vector(el + 1, U[\text{n} + 1, 2]) :$
> $plot3 := pointplot(\langle h|Y\rangle, symbolsize = 15, color = ["Blue"]) :$
> $plot(\langle h|Y\rangle, symbolsize = 15, color = ["Blue"]);$



Global Mass and Stiffness Matrix

> $M := \left(\frac{h[2]}{6}\right).Matrix(6, [[2, 1, 0, 0, 0, 0], [1, 4, 1, 0, 0, 0], [0, 1, 4, 1, 0, 0], [0, 0, 1, 4, 1, 0], [0, 0, 1, 4, 1]], [0, 0, 0, 0, 1, 4]])$:

> $K := Matrix(6, [[5, -5, 0, 0, 0, 0], [-5, 10, -5, 0, 0, 0], [0, -5, 10, -5, 0, 0], [0, 0, -5, 10, -5, 0], [0, 0, 0, -5, 10, -5], [0, 0, 0, 0, -5, 10]])$:

Calculations - FEM

> $v0 := -M^{-1}.K.d0$:

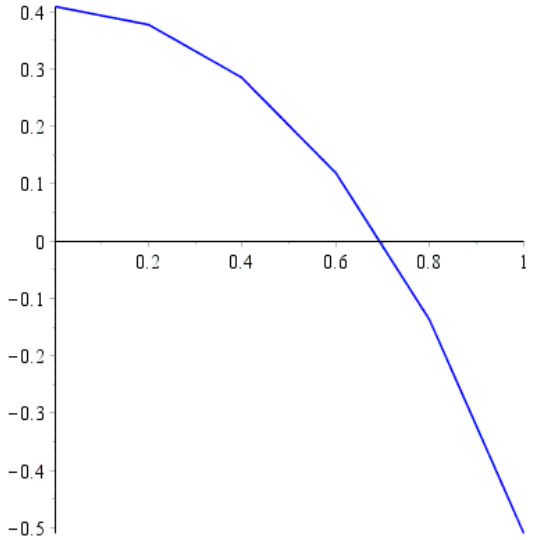
> $U := beuler(M, K, d0, n, dt, el, v0, a)$:

> $with(plots, pointplot)$:

> $Y := Vector(el + 1, U[n + 1, 2])$:

> $plot1 := pointplot(\langle h|Y\rangle, symbolsize = 15, color = ["Blue"])$:

> $plot(\langle h|Y\rangle, symbolsize = 15, color = ["Blue"])$;



Calculations - Exact dt=0.4

> $$Exact\_04 := x \rightarrow sum\left(\left(\frac{4}{3.14^{1}} \cdot \frac{(-1)^{m}}{(2\,\mathrm{m}+1)^{1}}\left(\frac{1}{\exp(1)^{\left(\frac{(3.14\cdot(2\,\mathrm{m}+1))^{2}}{4}\right)\cdot 0.4}} \cdot \cos\left(\frac{3.14}{2} \cdot (2\,\mathrm{m}+1) * \mathrm{x}\right)\right)\right), m = 0..1\right) : \% = value(\%);$$

$$\left(x \rightarrow \sum_{m=0}^{1} \frac{4\,(-1)^{m}\cos\left(\frac{3.14\,(2m+1)\,x}{2}\right)}{3.14\,(2m+1)\,(e)^{\frac{3.14^{2}\,(2m+1)^{2}\cdot 0.4}{4}}}\right) = \left(x \rightarrow \sum_{m=0}^{1} \frac{4\,(-1)^{m}\cos\left(\frac{3.14\,(2m+1)\,x}{2}\right)}{3.14\,(2m+1)\,(e)^{\frac{3.14^{2}\,(2m+1)^{2}\cdot 0.4}{4}}}\right)$$

> $with(plots)$:
> $plot2 := plot(Exact\_04(x), x = 0..1)$:
> $plot(Exact\_04(x), x = 0..1)$:
> $display([plot1, plot2],)$:

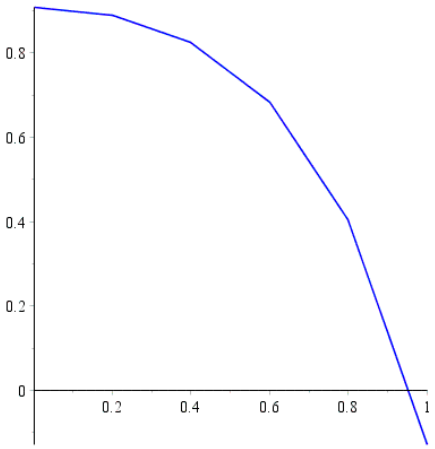
Time steps = 0.1

> $n := 2$:
> $Time := 0.2$:
> $$dt := \left(\frac{\mathrm{Time}}{\mathrm{n}}\right)\cdot(1);$$

$$dt := 0.1000000000$$

> $U := beuler(M, K, d0, n, dt, el, v0, a)$:
> $with(plots, pointplot)$:

> $Y := Vector(el + 1, U[\mathrm{n}+1, 2])$:
> $plot3 := pointplot(\langle h|Y\rangle, symbolsize = 15, color = ["Blue"])$:
> $plot(\langle h|Y\rangle, symbolsize = 15, color = ["Blue"])$;



Calculations - Exact dt=0.04

> $$Exact\_004 := x \rightarrow sum\left(\left(\frac{4}{3.14^{1}} \cdot \frac{(-1)^{m}}{(2\,m+1)^{1}}\left(\frac{1}{\exp(1)^{\left(\frac{(3.14\cdot(2\,m+1))^{2}}{4}\right)\cdot 0.04}} \cdot \cos\left(\frac{3.14}{2} \cdot (2\,m+1) * x\right)\right)\right), m = 0..1\right) : \% = value(\%);$$

$$\left(x \rightarrow \sum_{m=0}^{1} \frac{4\,(-1)^{m}\cos\left(\frac{3.14\,(2\,m+1)\,x}{2}\right)}{3.14\,(2\,m+1)\,(e)^{\frac{3.14^{2}\,(2\,m+1)^{2}\cdot 0.04}{4}}}\right) = \left(x \rightarrow \sum_{m=0}^{1} \frac{4\,(-1)^{m}\cos\left(\frac{3.14\,(2\,m+1)\,x}{2}\right)}{3.14\,(2\,m+1)\,(e)^{\frac{3.14^{2}\,(2\,m+1)^{2}\cdot 0.04}{4}}}\right)$$

> $with(plots) :$
> $plot8 := plot(Exact\_004(x), x = 0..1) :$
> $plot(Exact\_004(x), x = 0..1) :$

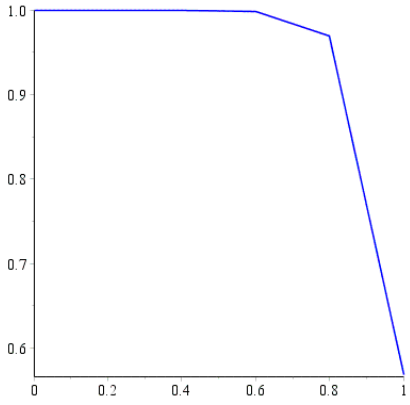
Time steps = 0.01

> $n := 2 :$
> $Time := 0.02 :$
> $$dt := \left(\frac{\text{Time}}{\text{n}}\right) \cdot (1);$$

$$dt := 0.01000000000$$

> $U := beuler(M, K, d0, n, dt, el, v0, a) :$
> $with(plots, pointplot) :$

> $Y := Vector(el + 1, U[\text{n} + 1, 2]) :$
> $plot9 := pointplot(\langle h|Y\rangle, symbolsize = 15, color = [\text{"Blue"}]) :$
> $plot(\langle h|Y\rangle, symbolsize = 15, color = [\text{"Blue"}]);$



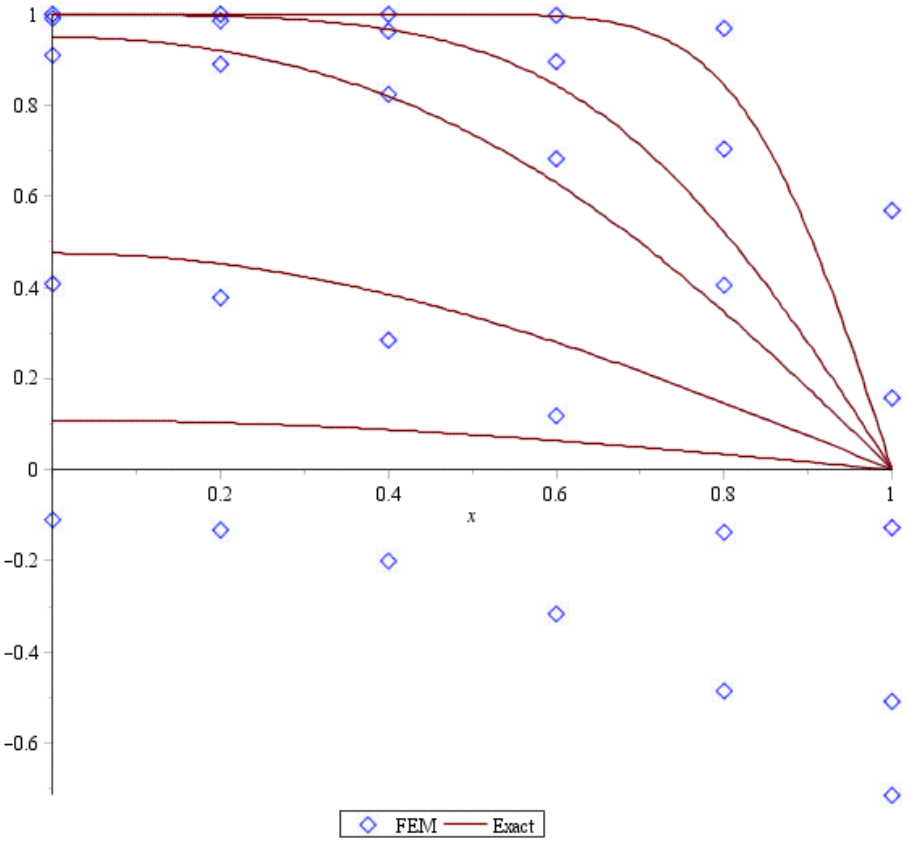
Calculations - Exact dt=0.01

>

> $$Exact\_001 := x \rightarrow sum\left(\left(\frac{4}{3.14^{1}} \cdot \frac{(-1)^{m}}{(2\,m+1)^{1}}\left(\frac{1}{\exp(1)^{\left(\frac{(3.14\cdot(2\,m+1))^{2}}{4}\right)\cdot 0.01}} \cdot \cos\left(\frac{3.14}{2}\cdot(2\,m+1)*x\right)\right)\right), m = 0..1\right) : \% = value(\%);$$

$$\left(x \rightarrow \sum_{m=0}^{1} \frac{4\,(-1)^{m}\cos\left(\frac{3.14\,(2\,m+1)\,x}{2}\right)}{3.14\,(2\,m+1)\,(e)^{\frac{3.14^{2}\,(2\,m+1)^{2}\cdot 0.01}{4}}}\right) = \left(x \rightarrow \sum_{m=0}^{1} \frac{4\,(-1)^{m}\cos\left(\frac{3.14\,(2\,m+1)\,x}{2}\right)}{3.14\,(2\,m+1)\,(e)^{\frac{3.14^{2}\,(2\,m+1)^{2}\cdot 0.01}{4}}}\right)$$

> $with(plots)$:
> $plot10 := plot(Exact\_001(x), x = 0..1)$:
> $plot(Exact\_001(x), x = 0..1)$:
> $display([plot1, plot2, plot3, plot4, plot5, plot6, plot7, plot8, plot9, plot10])$;



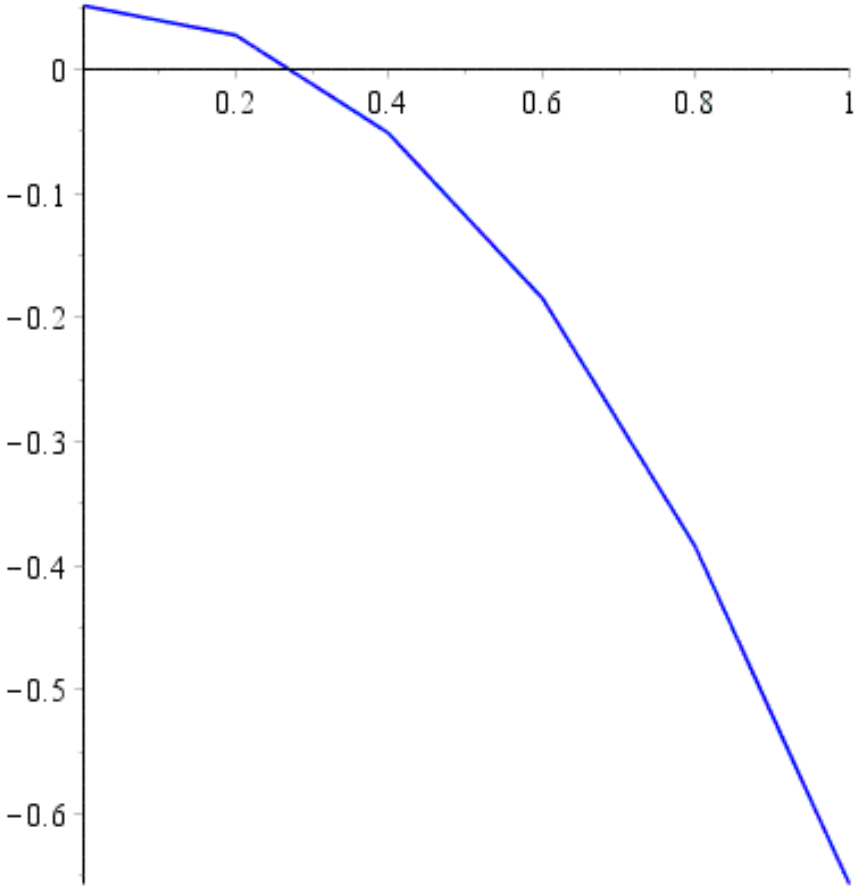
Time steps = 0.75

> $n := 2$:
> $Time := 1.5$:
> $dt := \left(\frac{\text{Time}}{\text{n}}\right) \cdot (1)$;

$$dt := 0.7500000000$$

> $U := beuler(M, K, d0, n, dt, el, v0, a)$:
> $with(plots, pointplot)$:

> $Y := Vector(el + 1, U[\text{n} + 1, 2])$:
> $plot11 := pointplot(\langle h|Y\rangle, symbolsize = 15, color = ["Blue"])$:
> $plot(\langle h|Y\rangle, symbolsize = 15, color = ["Blue"])$;



Calculations - Exact dt=0.75

>
> $$Exact\_075 := x \rightarrow sum\left(\left(\frac{4}{3.14^{1}} \cdot \frac{(-1)^{m}}{(2\,\text{m}+1)^{1}}\left(\frac{1}{\exp(1)^{\left(\frac{(3.14\cdot(2\,\text{m}+1))^{2}}{4}\right)\cdot 0.5}} \cdot \cos\left(\frac{3.14}{2} \cdot (2\,\text{m}+1) * \text{x}\right)\right)\right), m = 0..1\right) : \% = value(\%);$$

$$\left(x \to \sum_{m=0}^{1} \frac{4\,(-1)^m \cos\left(\frac{3.14\,(2m+1)\,x}{2}\right)}{3.14\,(2m+1)\,(e)^{\frac{3.14^2\,(2m+1)^2 \cdot 0.5}{4}}} \right) = \left(x \to \sum_{m=0}^{1} \frac{4\,(-1)^m \cos\left(\frac{3.14\,(2m+1)\,x}{2}\right)}{3.14\,(2m+1)\,(e)^{\frac{3.14^2\,(2m+1)^2 \cdot 0.5}{4}}} \right)$$

```
> with(plots) :
> plot12 := plot(Exact_075(x), x = 0 ..1) :
> plot(Exact_075(x), x = 0 ..1) :
> display([plot11, plot12]);
```